

Incompressible Fields in Riemannian Manifolds

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ABSTRACT. Incompressible fields are of special importance in electrodynamics, fluid mechanics, and quantum mechanics. We shall derive a few expressions for such fields in a Riemannian manifold, and show how to generate an incompressible field from an arbitrary set of scalar differentiable functions. The concept of compressibility removing factors of an arbitrary vector field is introduced and utilized to obtain from an arbitrary vector field an incompressible one that has the same vector surfaces as the original field. A general expression for compressibility removing factors of a vector field is derived. The method is applied to central fields.

1. INTRODUCTION

The equation which expresses that a vector field L has a zero divergence, $\text{div} L = 0$, appears in a number of areas of physics such as electrodynamics, fluid mechanics, and quantum mechanics. In fact two of Maxwell's equations, namely $\text{div } \vec{E} = \rho/\epsilon_0$ and $\text{div } \vec{H} = 0$, where \vec{E} and \vec{H} are the electric and magnetic fields respectively, ρ is the charge density and ϵ_0 is the permittivity of the medium, reduce in the free space to the form we have mentioned. Also the continuity equation $\text{div } \vec{j} + \partial\rho/\partial t = 0$ [4] in which \vec{j} is the current density of some flow of matter or charge, ρ is its volume density and t is the time, is reducible to this form when the volume density does not change with time. More recently [6], it was shown that the incompressible vector fields in a manifold, single out a class of quantum momentum observables that exhibit the features of the Cartesian momentum operators in a Euclidean space. In fact it was shown that the eigenfunctions of any member of this class is a direct generalization of De Broglie waves. The solution of the equation $\text{div} L = 0$ in the three dimensional Euclidean space is well known. It is given in terms of a C^2 vector field \vec{A} , by $\vec{L} = \nabla \times \vec{A}$. In this work

- (1) We shall find a few expressions for the solutions of this equation in an n -dimensional Riemannian manifold.
- (2) Show that every $(n-1), C^2$ functions determine an incompressible field.

(3) Introduce the concept of a compressibility removing factor of a vector field, construct some methods for determining such factors and find the most general expression of such factors.

(4) Show that the quotient of two compressibility removing factors of a field is a vector surface of that field.

(5) Apply the results we have obtained to central fields and centrally symmetric fields.

We shall use tensor notations in aim to simplify the calculations and formulae obtained. However the reader must be warned that the expressions we get for incompressible vector fields, though valid in any coordinate system, have no tensorial character. i.e. they do not transform as a contravariant vector.

2. THE FIELD EQUATION

Let M_n be a Riemannian manifold with (x^1, \dots, x^n) as a chart in which the metric takes the form $ds^2 = g_{ij}dx^i dx^j$, and let $g = \det(g_{ij})$. Let L be a C^1 vector field on M_n which has in the given chart the form $L = \xi^i \partial / \partial x^i$. The field L is said to be incompressible if its divergence

$$\operatorname{div} L = g^{-1/2} (g^{1/2} \xi^i)_{,i} \quad (1)$$

is zero [1]. According to (1) the field L is incompressible if

$$(\sqrt{g} \xi^i)_{,i} = 0. \quad (2)$$

The incompressible field equation (2) may be written as

$$A^i_{,i} = 0 \quad \text{where} \quad A^i = \sqrt{g} \xi^i.$$

Equation (2) admits obvious solutions such as

$$\xi^r = g^{-1/2}, \quad \xi^j = 0 \quad (j = 1, \dots, n; j \neq r), \quad (3)$$

where r is a fixed index chosen arbitrarily from the set $\{1, 2, \dots, n\}$. Since the field equation (2) is linear in the unknown functions ξ^j (or A^j) all linear combinations in the solutions (3) are also solutions of the field equation. We shall see also that the commutator of two incompressible fields is also incompressible, i.e., the set of incompressible fields in M_n forms a Lie algebra.

In the case of an one dimensional manifold M_1 which is coordinated by x so that $ds^2 = g(x) dx^2$, the most general solution of (2) is $L = \frac{c}{\sqrt{g}} \frac{d}{dx}$, where c is an arbitrary constant. However M_1 is certainly Euclidean and the metric in M_1 may be reduced to the standard form $ds^2 = dX^2$, by the transformation $X = \int_{x_0}^x \sqrt{g} dx$. The

incompressible field in the new coordinate X takes the form $L = c d/dX$ which is an infinitesimal motion of M_1 . When M_1 is isometric to the real line \mathbb{R} or to the one dimensional circle S^1 , the field L generates the symmetry group $U(X) = X + t, (t \in \mathbb{R})$, of motions of M_1 .

In a two-dimensional manifold M_2 , a field L is incompressible if $A_{,1}^1 + A_{,2}^2 = 0$, a condition which is satisfied if and only if the ordinary differential equation $A^2 dx^1 - A^1 dx^2 = 0$ is exact; which in turn is equivalent to the existence of a function B of class C^2 on M_2 such that $B_{,1} = -A_2$ and $B_{,2} = A_1$. It follows that the most general form of an incompressible field in M_2 is of the form

$$L = \frac{1}{\sqrt{g}} \left(\frac{\partial B}{\partial x^2} \frac{\partial}{\partial x^1} - \frac{\partial B}{\partial x^1} \frac{\partial}{\partial x^2} \right). \quad (4)$$

It is tempting to generalize the last result in a straight forward manner to an M_n . To achieve that we start with an arbitrary C^n function B , and take $A^i = \sqrt{g} \xi^i$ given by

$$A^1 = c_1 B_{,23\dots n}, \quad A^2 = c_2 B_{,13\dots n}, \dots, \quad A^n = c_n B_{,12\dots n-1}, \quad (5)$$

where c_i are arbitrary constants such that $\sum_{i=1}^n c_i = 0$. We have then

$$A_{,i}^i = \sum_{i=1}^n c_i B_{,12\dots n} = 0,$$

which proves that the field L , as given by (5), is incompressible. If all components of L are non-zero we may satisfy $\sum c_i = 0$ by

$$c_i = (-i)^{i-1} \binom{n-1}{i-1},$$

where no summation over i is implied in the last equation. When the dimension of the manifold is even one may choose an alternative solution $c_i = (-1)^{i-1}$. This method however, determines only a narrow class of incompressible fields in M_n . i.e., if $L = \xi^i \partial / \partial x^i$ is incompressible then there may not exist a function B such that (5) is satisfied. The following theorem singles out the case in which an incompressible field L may be expressed by (5)

Theorem 1. *An incompressible field $L = \xi^i \partial / \partial x^i$ can be expressed by the form (5) if and only if there exist constants $c_i (i = 1, \dots, n)$ such that $\sum c_i = 0$ and*

$$\frac{1}{c_1} A_{,1}^1 = \frac{1}{c_2} A_{,2}^2 = \dots = \frac{1}{c_n} A_{,n}^n. \quad (6)$$

If for some index r , $A_{,r}^r = 0$ then we take $c_r = 0$ and exclude the corresponding ratio from the equalities (6).

For proof see appendix 1.

3. GENERAL EXPRESSIONS FOR AN INCOMPRESSIBLE FIELD

3.1. A First Expression:. This results from taking $(n - 1)$ components ξ^j of L as arbitrary functions of class C^1 and calculating the remaining component ξ^r ($r \neq j$) by the field equation (2); it is given by

$$\sqrt{g}\xi^r = - \sum_{j \neq r} \int (\sqrt{g}\xi^j)_{,j} dx^r + h^r, \quad (7)$$

where h^r is an arbitrary function of class C^1 independent of x^r . Choosing every component ξ^j as the partial derivative with respect to x^r of an arbitrary function B^j of class C^2 , we write (7) in the equivalent form

$$\sqrt{g}\xi^j = B^j_{,r} \ (j \neq r), \quad \sqrt{g}\xi^r = - \sum_{j \neq r} B^j_{,j}.$$

We note that the arbitrary function h^r has been absorbed in $B^j_{,r}$.

3.2. A second Expression:. Let $e^{i_1 \dots i_n}$ be the totally antisymmetric unit tensor [3]

$$\begin{aligned} e^{i_1 \dots i_n} &= 1 \text{ if } (i_1 \dots i_n) \text{ is an even permutation of } (1, 2, \dots, n) \\ &= -1 \text{ if } (i_1 \dots i_n) \text{ is an odd permutation of } (1, 2, \dots, n) \\ &= 0 \text{ if two indices are equal,} \end{aligned}$$

and let $B_{i_1 \dots i_{n-2}}$, where $i_1, \dots, i_{n-2} \in \{1, \dots, n\}$ and $i_1 < i_2 < \dots < i_{n-2}$, be arbitrary $\binom{n}{2}$ functions of class C^2 on M_n .

Theorem 2. *The field*

$$L = \frac{1}{\sqrt{g}} e^{i_1 \dots i_n} B_{i_1 \dots i_{n-2}, i_{n-1}} \partial / \partial x^{i_n} \quad (8)$$

is incompressible.

Proof: See appendix A2

3.3. A Third Expression:. Let

$$\phi_k(x^1, \dots, x^n) \quad k \in [1, n-1] \quad (9)$$

be $(n - 1)$ arbitrary C^2 functions on M_n that are functionally independent. i.e.

$$\text{rank}(\partial \phi^k / \partial x^j) = n - 1$$

Theorem 3. *The field*

$$L = \frac{1}{\sqrt{g}} e^{i_1 \dots i_n} \phi_{1,i_1} \phi_{2,i_2} \dots \phi_{n-1,i_{n-1}} \frac{\partial}{\partial x^{i_n}} \quad (10)$$

in M_n is incompressible.

Proof: See appendix A3.

Remark 1. *The expression (10) may be written as*

$$L = \frac{1}{\sqrt{g}} \begin{vmatrix} \phi_{1,1} & \phi_{1,2} & \dots & \phi_{1,n} \\ \phi_{n-1,1} & \phi_{n-1,2} & \dots & \phi_{n-1,n} \\ \partial/\partial x^1 & \partial/\partial x^2 & \dots & \partial/\partial x^n \end{vmatrix} \quad (11)$$

where the determinant must be expanded in a way that the minor of $\partial/\partial x^i$ is the component of ξ^i .

Remark 2. *Had the functions (9) were functionally dependent then*

$$\text{rank}(\partial\phi_k/\partial x^i) < n - 1, \quad k \in [1, n - 1], \quad i \in [1, n],$$

and the determinant of every $(n - 1) \times (n - 1)$ sub-matrix of the last matrix vanishes. However these determinants, apart from a possible difference in sign, are precisely the components of L . Hence L vanishes.

Theorem 4. *Each of the surfaces*

$$\phi_k(x^1, \dots, x^n) = c_k \quad k \in [1, n - 1] \quad (12)$$

is a vector surface of the field (10), and hence the intersection of these surfaces are the field's lines.

Proof. Let the field L , as given by the expression (11), acts on ϕ_k . It is apparent that the determinant on the right hand-side vanishes, on the account of its k -th and last rows are equal.

Remarks and Examples

1. We have seen that in M_2 there exists a unique expression of an incompressible field, namely (4). Applying any of the last expressions we get (4)

2. When we apply the second expression to M_3 we get

$$\begin{aligned} L &= g^{-1/2} e^{ijk} B_{i,j} \partial / \partial x^k \\ &= g^{-1/2} [(B_{2,3} - B_{3,2}) \partial / \partial x^1 + (B_{3,1} - B_{1,3}) \partial / \partial x^2 + (B_{1,2} - B_{2,1}) \partial / \partial x^3] \end{aligned}$$

where B_1, B_2 and B_3 are arbitrary functions of class C^2 . This expression is equivalent to the familiar expression $\vec{L} = \nabla \times \vec{B}$ of an incompressible field in a rectangular coordinates in a Euclidean manifold M_3 .

Applying the third expression in M_3 we get

$$L = g^{-1/2} e^{ijk} \phi_{,i} \psi_{,j} = g^{-1/2} \begin{vmatrix} \phi_{,1} & \phi_{,2} & \phi_{,3} \\ \psi_{,1} & \psi_{,2} & \psi_{,3} \\ \partial / \partial x^1 & \partial / \partial x^2 & \partial / \partial x^3 \end{vmatrix}.$$

This reduces when M_3 is Euclidean and the coordinates are Cartesian to $\vec{L} = \nabla \phi \times \nabla \psi$.

3. The expression (10) of an incompressible field L is equivalent to obtaining the system of ordinary differential equations, whose integrals $\phi_{,k}$, $k \in [1, n-1]$ are given, through taking the differentials $d\phi_k = 0$, solving for $(n-1)$ of the dx^i , and eventually symmetrizing the system of ordinary differential equation that has been obtained.

4. REDUCTION OF A VECTOR FIELD TO AN INCOMPRESSIBLE ONE

Let $K = \eta^i \partial / \partial x^i$ be a C^1 vector field on M_n , and $\mu(x^1, \dots, x^n)$ be a C^1 function on M_n . The function μ is said to be a compressibility removing factor (*CRF* for short) of the vector field K , or it removes the compressibility of K , if the vector field μK is incompressible. If μ removes the compressibility of K we say that μK reduces the field K . A function μ is a *CRF* of K if and only if it satisfies the equation $\text{div}(\mu K) = 0$, which is equivalent to $\mu \text{div} K + K\mu = 0$. This equation can be deduced directly, or by using the properties of *div* [1]. The equation of *CRF*

$$K\mu = -\mu \text{div} K \quad (13)$$

is a partial differential equation of Lagrange type. The subsidiary system associated with this first order linear equation [5] is

$$\frac{dx^1}{\eta^1} = \dots = \frac{dx^n}{\eta^n} = \frac{d\mu}{-\mu \text{div} K} \quad (14)$$

We shall denote the set of incompressible fields which reduces K by $L(\mu, K)$.

Theorem 5. *Each member of the set $L(\mu, K)$ has the same vector surfaces of K .*

Proof. It follows from observing that the vector surfaces of all the fields μK , as well as the field K , are determined by the first n ratios in (14). Or from noting that $\mu K \phi = 0 \Leftrightarrow K \phi = 0$.

Remark 3. The last theorem is applicable to every vector field of the form λK , where λ is any function that is not zero. This is true whether or not λ reduces K .

Let

$$\phi_k(x^1, \dots, x^n) = c_k \quad k \in [1, n-1] \quad (15)$$

be functionally independent integrals of (14) resulting from the first n ratios. The most general arbitrary integral associated with the field K is of the form $f(\phi_1, \dots, \phi_{n-1})$, where f is an arbitrary function of the ϕ_k . Hence every vector surface of the field K has the form $f(\phi_1, \dots, \phi_{n-1}) = 0$

Theorem 6. If the CRFs μ_1 and μ_2 of the field K are linearly independent then μ_1/μ_2 is an integral of K .

Proof. By the equation of CRFs (13), $\mu_2 K \mu_1 = \mu_1 K \mu_2$, and hence

$$K(\mu_1/\mu_2) = \mu_2^{-2}(\mu_2 K \mu_1 - \mu_1 K \mu_2) = 0$$

Therefore μ_1/μ_2 is an integral of the subsidiary system associated with K , and $\mu_1/\mu_2 = c$, where c is an arbitrary constant, is a vector surface of K .

Theorem 7. If μ is a CRF of the field K then all its CRFs are given by

$$M = \mu f(\phi_1, \dots, \phi_{n-1}), \quad (16)$$

where f is a C^1 arbitrary function in the functions ϕ_k .

Proof. Let M be a CRF of the field K . By theorem (4.2) M/μ is an integral of the subsidiary system associated with the field K , and hence it has the form

$$M/\mu = f(\phi_1, \dots, \phi_{n-1}),$$

which completes the proof.

One can verify easily that the equation of CRFs (13) is satisfied by M given by (16):

$$\begin{aligned} (\mu f) \operatorname{div} K + K(\mu f) &= \mu f \operatorname{div} K + f K \mu + \mu K f \\ &= f.(\mu \operatorname{div} K + K \mu) + 0 \text{ (because } K f = 0). \end{aligned}$$

By (13) the right hand side vanishes.

It follows from theorem (7) that if K is incompressible then every CRF of K is of the form $M = f(\phi_1, \dots, \phi_{n-1})$.

We have seen in section 2 that a field $L = \xi^1 \partial / \partial x^1 + \xi^2 \partial / \partial x^2$ is incompressible if and only if the equation

$$\sqrt{g} \xi^2 dx^1 - \sqrt{g} \xi^1 dx^2 = 0 \quad (17)$$

is exact. It follows that μ is a CRF of L if and only if μ is an integrating factor of (17), i.e. if and only if $\mu \sqrt{g}$ is an integrating factor of $\xi^2 dx^1 - \xi^1 dx^2 = 0$.

We finally mention that, although there exists an infinite family $L(\mu, K)$ of fields which have the same vector surfaces as K does, the integral curves of these fields are distinguishable. Indeed, the differential equations of the integral curves of a field μK , namely

$$dx^i = \mu \eta^i dt,$$

depend on the function μ .

5. METHOD FOR DETERMINING $CRFs$

1. If it were possible to find an integral of the system (14) that contains μ then on solving for μ we get one of the required factors.

We shall assume in what follows that we have available $(n - 1)$ integrals of the field K which we denote by $\phi_k (k = 1, \dots, n - 1)$.

2. Since the integrals ϕ_k of K are also integrals to every incompressible field $L = \mu K$, we may find one of the fields $L = \mu K$ say $L = \xi^i \partial / \partial x^i$ using the expression (10) of an incompressible field in terms of its vector surfaces. Assuming that the component η^r of the field K is not zero, then

$$\mu = \xi^r / \eta^r = e^{i_1 \dots i_{n-1} r} \phi_{1, i_1} \dots \phi_{n-1, i_{n-1}} / \eta^r \quad (18)$$

is one of the required factors.

3. Assuming that $rank(\partial \phi_j / \partial x^k) = n - 1$ where $k, j \in [1, n - 1]$, we may solve (15) for (x^1, \dots, x^{n-1}) in the form

$$x^k = x^k(c_1, \dots, c_{n-1}, x^n) \quad k \in [1, n - 1]. \quad (19)$$

If we could obtain the solution (19) we substitute in (14), separate the variables and integrate to find

$$\mu = c_n e^{-\int \frac{div L}{\eta^n} dx^n} = c_n e^{-F(c_1, \dots, c_{n-1}, x^n)}, \quad (20)$$

where $x^k, k \in [1, n - 1]$, are substituted for in $F = \int \frac{div L}{\eta^n} dx^n$ from (19). Substituting for c_1, \dots, c_{n-1} in (20) from (19) we get the n -th integral of (14)

$$\mu(x^1, \dots, x^n) = f(\phi_1, \dots, \phi_{n-1}) e^{-F(\phi_1, \dots, \phi_{n-1}, x^n)},$$

which is also the general form of a CRF .

6. EXAMPLE - CENTRAL FIELDS

Let $M_n \equiv E_3$ be the three dimensional Euclidean space with Cartesian coordinates $\vec{r} = (x^1 = x, x^2 = y, x^3 = z)$. We shall say that the field $K = \eta^i \partial / \partial x^i$ is central if its lines are straight lines through the origin; and centrally symmetric if it can be reduced to the form $K = \eta(r) \partial / \partial r$, where $r = |\vec{r}|$.

Consider the centrally symmetric field

$$K = x \partial / \partial x + y \partial / \partial y + z \partial / \partial z. \quad (21)$$

Since $\text{div} K = 3$, the equation of *CRFs* is

$$K\mu + 3\mu = 0. \quad (22)$$

The subsidiary system associated with (22) is

$$dx/x = dy/y = dz/z = -d\mu/3\mu. \quad (23)$$

From (23) we deduce that each of the functions $\mu_1 = x^{-3}, \mu_2 = y^{-3}, \mu_3 = z^{-3}$ is a *CRF* of K . It follows that $\mu_1/\mu_2 = y^3/x^3, \mu_1/\mu_3 = z^3/x^3$ are two integrals of the subsidiary system associated with K , and hence every vector surface of K is of the form

$$f(y/x, z/x) = 0. \quad (24)$$

According to (16) the general form of a *CRF* is

$$\mu = x^{-3} f(y/x, z/x). \quad (25)$$

We may choose f so that we get the *CRFs* $\mu_4 = 1/xyz$ or $\mu_5 = (x^2 + y^2 + z^2)^{3/2}$. Every vector surface of the fields $\mu_i K (i = 1, \dots, 5)$ is given by (24), which shows that the lines of all these fields are straight lines through the origin, and hence these fields are central. The field K in spherical coordinates (r, ϕ, θ) is written as $K = r \partial / \partial r$ which shows that it is centrally symmetric, and that the product of K by any μ_i , except μ_5 , is not centrally symmetric; whereas $\mu_5 K = r^{-2} \partial / \partial r$ is centrally symmetric. It is interesting to notice that, apart from a multiplicative constant, $r^{-2} \partial / \partial r$ is the only incompressible centrally symmetric field in the Euclidean space E_3 . It is also proved easily that the most general expression for a centrally symmetric incompressible field in E_n is $r^{1-n} \partial / \partial r$, where it is understood of course that the space has been coordinated with a system of spherical coordinates.

Another way for obtaining a *CRF* of the field (21) is to start with the general integral (24) and construct an incompressible field $L = \xi^i \partial / \partial x^i$ by formula (10). The first component of L is

$$\xi^1 = e^{ij1} \phi_{,i} \psi_{,j} = \phi_{,3} \psi_{,2} - \phi_{,2} \psi_{,3} = x^{-2},$$

where $\phi = y/x$, $\psi = z/x$. By (18) $\mu = \xi^1/\eta^1 = x^{-3}$ is a *CRF*, which was obtained earlier, using a different method.

The integral curves of K are given by

$$dx^i = x^i dt \quad (i = 1, 2, 3).$$

Integrating the last system we get $x^i = x_0^i e^t$, where $x^i(t=0) = x_0^i$. The parametric equation of the integral curves is $\vec{r} = \vec{r}_0 e^t$.

The integral curves of the incompressible centrally symmetric field $r^{-2}\partial/\partial r$ are

$$r = (3t + r_0^3)^{-1/3}, \quad \phi = \phi_0, \quad \theta = \theta_0. \quad (26)$$

Since $r \geq 0$ we have $t \geq -r_0^3/3$. When $t \rightarrow -r_0^3/3$, $r \rightarrow 0$, and when $t \rightarrow +\infty$, $r \rightarrow +\infty$. In other words every point moves under the action of the transformation (26) in a straight line ($\phi = \phi_0, \theta = \theta_0$) ensuing from the origin when $t \rightarrow -r_0^3/3$ to occupy (r_0, ϕ_0, θ_0) at $t = 0$, and goes to infinity as $t \rightarrow +\infty$.

7. A NOTE ON THE ALGEBRAIC STRUCTURE

We know that the set of all vector fields in M_n , named by Γ , form a Lie algebra with respect to addition, multiplication by a scalar and the commutator operation. If $L = \xi^i \partial/\partial x^i$ and $L_1 = \eta^k \partial/\partial x^k$, then

$$\begin{aligned} L + L_1 &= (\xi^i + \eta^i) \partial/\partial x^i, \quad cL = (c\xi^i) \partial/\partial x^i, \\ [L, L_1] &= (\xi^i \eta_{,i}^k - \eta^i \xi_{,i}^k) \partial/\partial x^k. \end{aligned}$$

It is clear that the postulates of a vector space are satisfied and that $[,]$ is a commutator on Γ . Let $\mathfrak{S} \subset \Gamma$ be the set of all incompressible fields in M_n . We shall prove that \mathfrak{S} is a sub-algebra of Γ . Indeed, if $L, L_1 \in \mathfrak{S}$ then

$$\text{div}(L + L_1) = \text{div}L + \text{div}L_1 = 0, \quad \text{div}(cL) = c \text{div}L = 0,$$

and hence \mathfrak{S} is a subspace of the vector space Γ . Furthermore [1]

$$\text{div}[L, L_1] = L \text{div}L_1 - L_1 \text{div}L,$$

and hence $\text{div}[L, L_1] = 0$, and \mathfrak{S} is a sub-algebra of Γ .

Let $\Pi \subset \Gamma$ be the set of all Killing fields [2] in M_n . Since the divergence of every Killing field vanishes [2], we have $\Pi \subset \mathfrak{S}$. If not empty, Π is a Lie algebra [2], and $\Pi \subset \mathfrak{S} \subset \Gamma$.

Appendix A1. Proof of Theorem (2.1)

Necessary conditions are obvious. To prove sufficiency we define a function

$$B = \frac{1}{c^1} \int A_{,1}^1 dx^1 dx^2 \dots dx^n$$

Then

$$c_1 B_{,23\dots n} = A^1, \quad c_2 B_{,13\dots n} = \left(\int A_{,2}^2 dx^1 dx^2 \dots dx^n \right)_{,13\dots n} = A^2 \text{ (by (6))}, \text{ etc...}$$

Appendix A2. Proof of Theorem (3.1)

By (1)

$$\sqrt{g} \operatorname{div} L = e^{i_1 \dots i_n} B_{i_1 \dots i_{n-2}, i_{n-1} i_n}$$

For every two values of the indices i_{n-1} and i_n there corresponds a unique set of values to the indices $\{i_1, \dots, i_{n-2}\}$ such that $i_1 < \dots < i_{n-2}$, and hence there corresponds to given values of i_{n-1} and i_n in the last sum two terms which are equal in absolute value:

$$\in (B_{i_1 \dots i_{n-2}, i_{n-1} i_n} - B_{i_1 \dots i_{n-2}, i_n i_{n-1}}) = 0$$

The multiplier \in is equal to $+1$ or -1 according to the permutation $(i_1 \dots i_n)$ is even or odd. It follows that the sum over all the values of i_{n-1} and i_n vanishes.

Appendix A3. Proof of Theorem (3.2)

We have

$$\begin{aligned} (\sqrt{g} \xi^{i_n})_{,i_n} &= (e^{i_1 \dots i_n} \phi_{1,i_1} \phi_{2,i_2} \dots \phi_{n-1,i_{n-1}})_{,i_n} \\ &= e^{i_1 \dots i_n} (\phi_{1,i_1 i_n} \phi_{2,i_2} \dots \phi_{n-1,i_{n-1}} + \dots + \phi_{1,i_1} \phi_{2,i_2} \dots \phi_{n-1,i_{n-1} i_n}) \\ &= \left(\sum_{i_1 i_n} \phi_{1,i_1 i_n} \sum_{i_2 \dots i_{n-2}} e^{i_1 \dots i_n} \phi_{2,i_2} \dots \phi_{n-1,i_{n-1}} \right) + \dots \\ &\quad + \left(\sum_{i_{n-1} i_n} \phi_{n-1,i_{n-1} i_n} \sum_{i_1 \dots i_{n-2}} e^{i_1 \dots i_n} \phi_{1,i_1} \dots \phi_{n-2,i_{n-2}} \right) \end{aligned}$$

We shall prove that the content of every bracket vanishes. For arbitrary fixed values of the indices $\{i_2 \dots i_{n-1}\}$ the first bracket contains two terms which are equal in absolute value and different in sign, and is written as

$$\in (\phi_{1,i_1 i_n} - \phi_{1,i_n i_1}) (\phi_{2,i_2} \phi_{3,i_3} \dots \phi_{n-1,i_{n-1}}) = 0$$

The other brackets vanish similarly, and the theorem is proved.

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